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# Precise FEM solution of a corner singularity using an adjusted mesh

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## SUMMARY

Within the framework of the finite element method problems with corner-like singularities (e.g. on the well-known L-shaped domain) are most often solved by the adaptive strategy: the mesh near the corners is refined according to the *a posteriori* error estimates. In this paper we present an alternative approach. For flow problems on domains with corner singularities we use the *a priori* error estimates and asymptotic expansion of the solution to derive an algorithm for refining the mesh near the corner. It gives very precise solution in a cheap way. We present some numerical results. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: FEM; singularity; refinement; error estimates

## INTRODUCTION

In papers [1, 2] we solved the problem with corner singularities by means of an adaptive refinement strategy based on *a posteriori* error estimates. In this paper we present an alternative approach to the adaptive mesh refinement. It is based on knowledge of the singularity near the corner. For steady Navier–Stokes equations we proved in Reference [3] that for nonconvex internal angles the velocities near the corners possess an expansion $u(\rho,\vartheta)=\rho^{\gamma}\varphi(\vartheta)+\cdots$ (+ smoother terms), where $\rho,\vartheta$ are local spherical co-ordinates. The local behaviour of the solution near the singular point is used to design a mesh which is adjusted to the shape

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of the solution. We show an example of a 2D mesh with quadratic polynomials for velocity. Then we use this adjusted mesh for the numerical solution of flow in the channel with corners.

## MODEL PROBLEM

We consider two-dimensional flow of a viscous, incompressible fluid described by the Navier–Stokes equations in a domain with corner singularity, cf. Figure 1.

Due to symmetry, we solve the problem only on the upper half of the channel.

Let us denote this domain $\Omega \subset \mathbb{R}^2$. The steady Navier–Stokes problem for the incompressible fluid consists in finding the velocity $\mathbf{v} = (v_1, v_2)$, and pressure $p$ defined in $\Omega$ and satisfying

$$(\mathbf{v} \cdot \nabla)\mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f} \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

together with boundary conditions on disjoint parts of the boundary $\Gamma_{\text{in}}$, $\Gamma_{\text{wall}}$ and $\Gamma_{\text{out}}$ (meaning, respectively, the inlet, the wall, and the outlet part)

$$\mathbf{v} = \mathbf{g} \quad \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{wall}} \tag{3}$$

$$\nu \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n} = 0 \quad \text{on } \Gamma_{\text{out}} \quad \text{('do nothing' boundary condition)} \tag{4}$$

We consider kinematic viscosity $\nu = 0.000025\,\text{m}^2/\text{s}$ and $v_{\text{in max}} = 1\,\text{m/s}$, which gives a maximum Reynolds number around 760. We do not consider volumetric loads $\mathbf{f} = (f_1, f_2)$.

## ALGORITHM DERIVATION

In Reference [3] we proved for the Stokes flow in axisymmetric tubes that for internal angle $\alpha = \frac{3}{2}\pi$, like in Figure 1, the leading term of the expansion of the solution for the velocity components is

$$v_l(\rho, \vartheta) = \rho^{0.54448374} \varphi_l(\vartheta) + \cdots, \quad l = 1, 2 \tag{5}$$

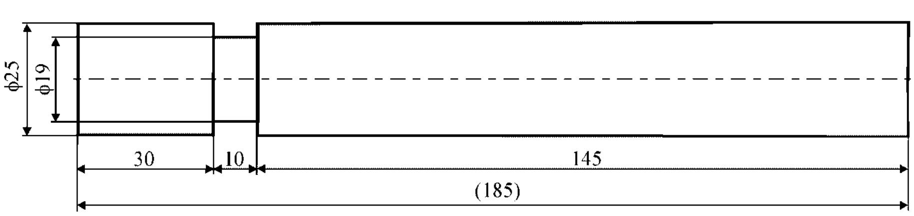


Figure 1. Geometry of the channel.

where $\rho$ is the distance from the corner, $\varphi_1$ is some smooth function of the angle $\vartheta$. The same expansion is known to apply to the plane flow, cf. Reference [4]. Similar results were proved for the Navier–Stokes equations. Differentiating by $\rho$ we observe $\partial v_i(\rho,\vartheta)/\partial\rho \to \infty$ for $\rho \to 0$.

*A priori* estimate of the finite element error is (cf. Reference [5])

$$\|\nabla(\mathbf{v}-\mathbf{v_h})\|_0 + \|p-p_h\|_0 \leqslant C\left[\left(\sum_T h_T^{2k}|\mathbf{v}|^2_{H^{k+1}(T)}\right)^{1/2} + \left(\sum_T h_T^{2k}|p|^2_{H^k(T)}\right)^{1/2}\right] \tag{6}$$

where $k=2$ for the Taylor–Hood elements we use. Taking into account expansion (5), we derived in Reference [3] the estimate

$$|\mathbf{v}|^2_{H^{k+1}(T)} \approx C\int_{r_T-h_T}^{r_T} \rho^{2(\gamma-k-1)}\rho\,\mathrm{d}\rho \approx C r_T^{2(\gamma-k)} \tag{7}$$

where $h_T$ is the diameter of the triangle $T$ of a triangulation $\mathscr{T}_h$, and $r_T$ is the distance of the element $T$ from the corner. Here $\gamma$ is the exponent at $\rho$ in expansion (5); $\gamma=0.54448374$ for internal angle $\alpha=\frac{3}{2}\pi$ (cf. Reference [3]).

Putting (7) into the *a priori* error estimate (6), we derive that we should guarantee

$$h_T^{2k}[-r_T^{2(\gamma-k)} + (r_T-h_T)^{2(\gamma-k)}] \approx h^{2k} \tag{8}$$

in order to get the error estimate of the order $O(h^k)$ and uniformly distributed on elements.

After simplifications we produced the approximate expression

$$h_T^{2k} r_T^{2(\gamma-k)} \approx h^{2k} \tag{9}$$

This led us in Reference [3] to an algorithm for generating the mesh near the corner in recursive form:

$$h_i = h\cdot(r_i)^{1-\gamma/k} \tag{10}$$

$$r_{i+1} = r_i - h_i, \quad i=1,2,\ldots,N \tag{11}$$

where $r_1$ is the distance of the large element from the corner. Using this algorithm we obtained satisfactory results. Some of them were presented in Reference [6].

At present, we have a program for computing the element sizes directly from expression (8) using Newton's method. This algorithm for mesh refinement is applied to the corner where the channel or tube suddenly decreases the diameter (forward step in Figure 1). We start with $r_1=0.25$ mm, $h=0.1732$ mm, $k=2$, $\gamma=0.5444837$. This corresponds to cca 3% of relative error on equidistributed elements. This way we get 14 diameters of elements, cf. Table I.

 

Table I. Resulting refinement.

| $i$ | $r_i$ (mm) | $h_i$ (mm) | $i$ | $r_i$ (mm) | $h_i$ (mm) |
|---|---|---|---|---|---|
| 1 | 0.25000 | 0.0600369 | 8 | 0.02311 | 0.0085813 |
| 2 | 0.18996 | 0.0480779 | 9 | 0.01453 | 0.0058366 |
| 3 | 0.14189 | 0.0379492 | 10 | 0.00869 | 0.0037980 |
| 4 | 0.10394 | 0.0294666 | 11 | 0.00489 | 0.0023392 |
| 5 | 0.07447 | 0.0224535 | 12 | 0.00255 | 0.0013434 |
| 6 | 0.05202 | 0.0167410 | 13 | 0.00121 | 0.0007042 |
| 7 | 0.03527 | 0.0121680 | 14 | 0.00050 | 0.0005042 |

## DESIGN OF THE MESH

In Reference [6], we showed that the best way to use data given by the algorithm is to design the mesh in correspondence with the polar co-ordinate system due to its usage in estimates. We continue this idea and design a two-dimensional mesh near the corner with singularity as can be seen in Figure 2. This 'patch' is connected to the whole computational mesh, cf. Figures 3 and 4.

## EVALUATION OF THE ERROR

To evaluate the error on elements we use now the modified absolute error computed using *a posteriori* error estimates, defined as

$$\mathscr{A}_m^2(v_1^h, v_2^h, p^h, \Omega_l) = \frac{|\Omega|\mathscr{E}^2(v_1^h, v_2^h, p^h, \Omega_l)}{|\overline{\Omega}_l|\|(v_1^h, v_2^h, p^h)\|_{V,\Omega}^2} \tag{12}$$

where $\mathscr{E}^2(v_1^h, v_2^h, p^h, \Omega_l)$ is the estimate of the error on element $l$, $|\Omega|$ is the area of the whole domain and $|\overline{\Omega}_l|$ is the mean area of elements obtained as $|\overline{\Omega}_l| = |\Omega|/n$. Here, $n$ means the number of all elements in the domain. More about the evaluation of the error can be found in References [1, 2].

## NUMERICAL RESULTS

In Figures 5–8 we present the graphical output of entities that characterize the flow in the channel. In Figure 5 there are the streamlines in the channel. Figure 6 with contours of velocity $v_y$ shows that the solution is satisfactorily smooth on the refined area. In Figures 7 and 8 we can observe how strong the singularity is, both for velocity and pressure (note that here the flow is from the right to the left, to have better view). In Figures 9 and 10 we show the errors on elements.

 

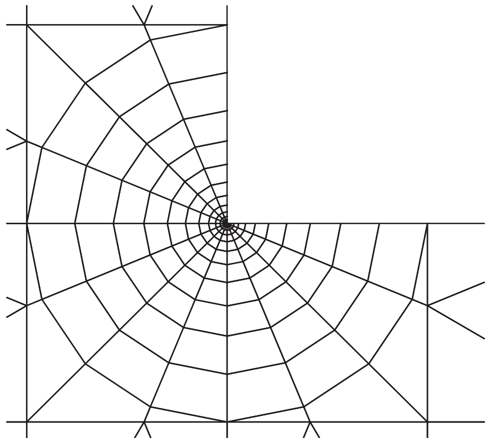

Figure 2. Refined detail of mesh.

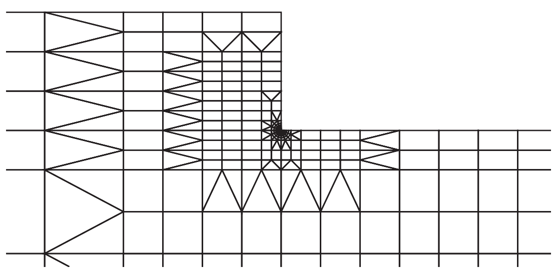

Figure 3. Computational mesh near the corner.

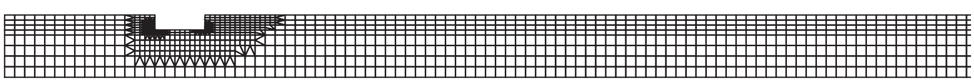

Figure 4. Whole computational mesh.

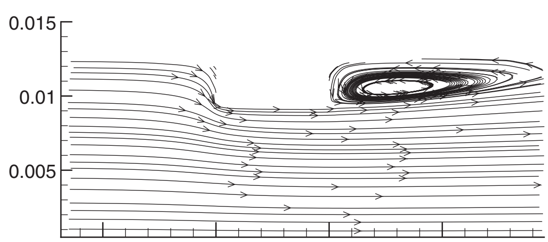


Figure 5. Streamlines.

 

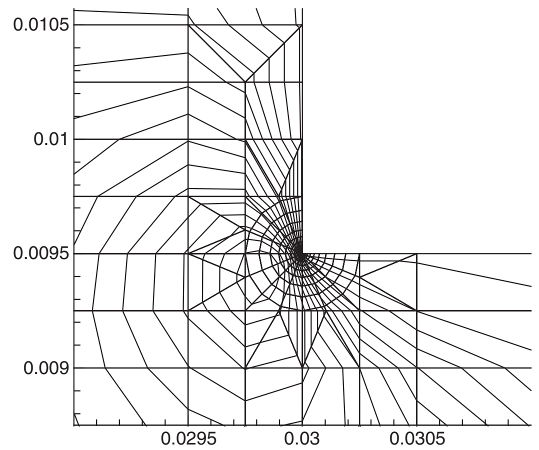


Figure 6. Isolines of velocity $v_y$.

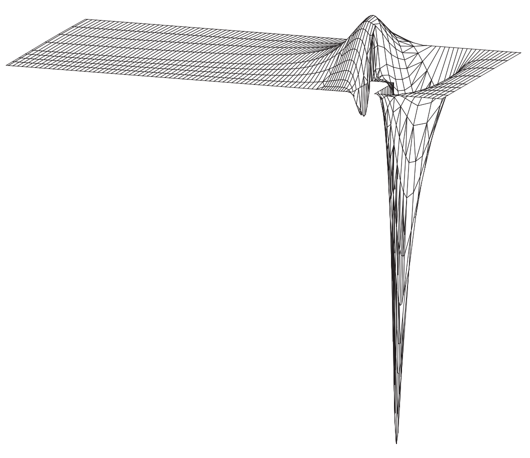

Figure 7. Velocity component $v_y$.

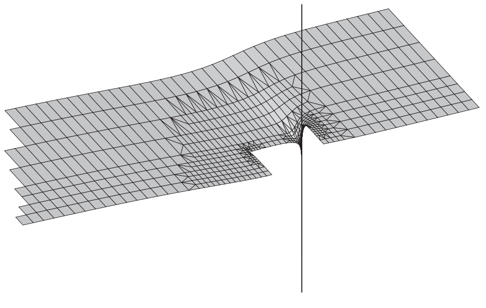

Figure 8. Pressure near the corner.

 

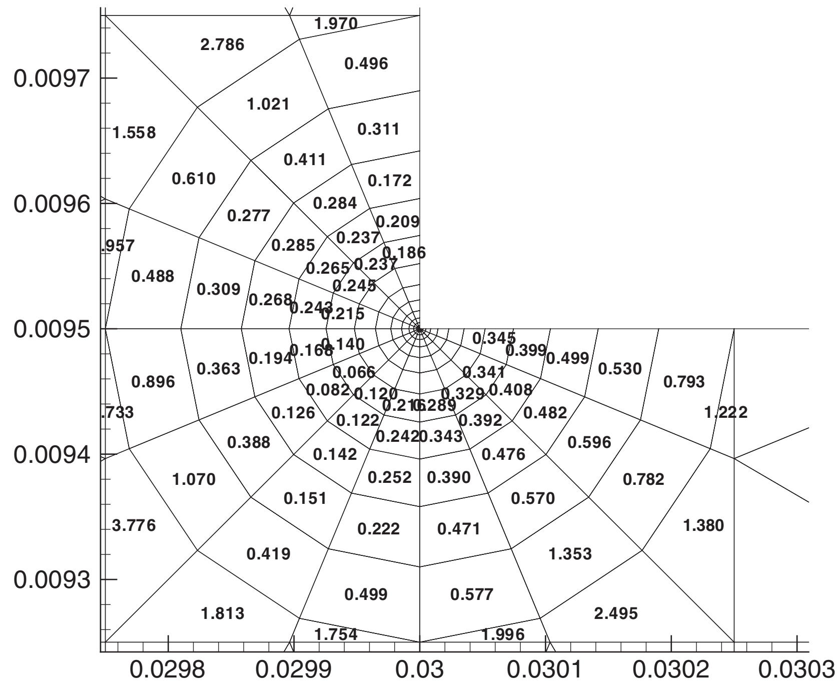


Figure 9. Errors on elements—whole refined area.

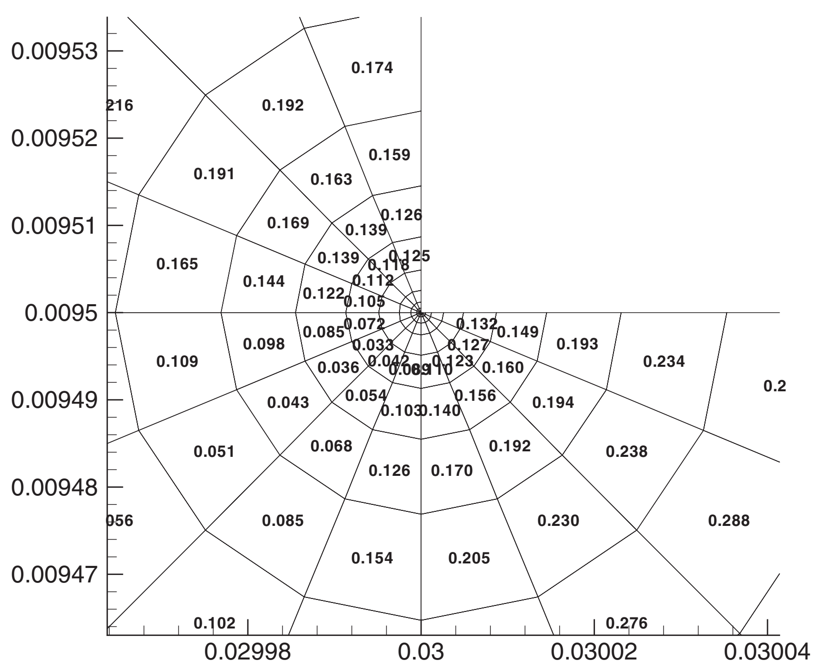


Figure 10. Errors on elements—detail.

## CONCLUSIONS

Presented results give satisfactory confirmation of the algorithm. The application of *a priori* error estimates of the finite element method for mesh refinement near the singularity is very efficient for our problem. This can be seen especially in the errors indicated on elements: the errors are distributed very uniformly.

The algorithm we derived is unique for the design of the mesh close to an internal angle of $\frac{3}{2}\pi$. Nevertheless, it admits to generate the mesh for other angles as well, in accordance with the parameter $\gamma$ which must be found for the respective angle. The approach in this paper is an alternative to the 'classical' one, using adaptive mesh refinement, which is still much more robust.

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